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A MIXTURE THEORY FOR ELASTIC LAMINATED COMPOSITES

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Abstract—A theory is developed which governs the dynamic response of a homogeneous, elastic, dispersive material. The material is used as a model of a two phase layered material in which each of the layers is isotropically elastic. The theory is derived from a general theory for all two phase periodic materials which was developed earlier[1].

The general theory was derived using the theory of mixtures. Dispersion is accommodated through the use of elastodynamic operators and for the layered material a micro model is used to establish the forms of the operators appropriate to the material. These specific operators are simplified by replacing them with truncated power series before introducing them into the equations of linear momentum. The theory for layered materials contains nineteen model constants and equations are developed from which these constants can be derived from the layer constants. The equations are derived partly using micro model analysis and partly by matching specific dynamic behaviors of the model and prototype. The ability with which the model predicts the dynamic response of the layered material is assessed in two ways. Both compare spectra reflecting the behavior of infinite trains of the principal kinds of waves. The first compares spectral lines from the model with those derived from the exact theory for layered materials. The second compares lines from the model with those obtained from experiments. Predictions from the model prove to be quite accurate.

1. INTRODUCTION

This paper is the second of three devoted to developing a mathematical model of a material that can be used to study the response of masonry walls to dynamic excitations. In formulating the prototype we incorporate the following characteristics. Masonry walls consist of two phases, blocks or bricks, and mortar; each phase is isotropically elastic; they are periodic in array and, because of the bond between phases, the walls disperse waves travelling through them.

In the first paper[1], construction of the model began with the development of a theory for a homogeneous material which replaces general two phase materials. The materials are general except that they must display a periodic character and each of the phases must be linearly elastic. The model material was constructed using the theory of mixtures and the reasons for this choice are listed in Ref.[1].

In what follows, Section 2 is devoted to a brief review and restatement of this theory, which consists of two types of equations. The first are the constitutive equations in which the stresses and strains are related by 78 constants, and the second are the equations of linear momentum. Dispersion is accommodated in the theory through so called elastodynamic operators which appear in the latter equations.

In Section 3 the general theory is modified for a specific periodic array, namely a layered material in which the phases appear as alternate plane layers. This geometry is chosen for two reasons; firstly, because it is close to the geometry displayed by masonry and, secondly, because there is a wealth of information about the dynamic response of such materials, both from experiments and predicted by an exact theory.

Both the constitutive equations and the equations of linear momentum are affected by this choice of geometry. The layered material displays hexagonal symmetry which reduces the number of independent constitutive constants from 78 to 15. The elastodynamic operators appear in the general theory in symbolic form needing a specific geometry for their formulation.

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These operators are constructed in Section 3 from a study of a micro model, or cell, of the layered material. This analysis is similar to one used by Biot[2, 3] for establishing a viscodynamic operator for a fluid-filled porous medium. The elastodynamic operators reflect the behaviors at the interfaces. We felt that as first constructed the operators would render the equations of linear momentum too complicated for realistic dynamic problems, so the final part of Section 3 is devoted to replacing them by simpler approximations. To this end the operators are expanded in power series of their argument, and the first three terms are retained. The resulting equations of linear momentum contain four constants to accommodate dispersion. The final theory which acts as a model for a two phase layered composite contains 19 model constants.

Section 5 is devoted to constructing equations relating these 19 model constants to the elastic constants of each of the two phases. The constants are adjusted so that the dynamic responses of model and prototype will match as extensively as possible. The wealth of information about the behavior of waves travelling in layered materials is extremely useful in this section. The characteristics of the dynamic behavior of layered materials are almost without limit, certainly far exceeding the 19 needed to establish the unknown constants. So the constants are not unique and will change according to which set of behaviors is chosen for matching. We discuss the choice at some length in Section 5 pointing out that the best set would be obtained using system identification in conjunction with response obtained from experiments. The method used here to derive the set of equations is, we think, the simplest. Some of the equations are found using micro model analysis, while the remainder by matching properties of spectral lines which reflect the behavior of infinite trains of the principal types of waves as predicted by the model and the exact theory for layered materials. With the capability of establishing the 19 model constants from the phase constants, the model is complete.

In the final section, Section 6, we make a preliminary assessment of the model material. We consider it preliminary since, in a third paper, we make a much more demanding, and perhaps more realistic, appraisal of the theory. In that paper we compare the behaviors of transient waves as they propagate in the model and as observed in experiments conducted on layered materials.

The assessment here is made by comparing spectral lines derived from the model with comparable lines from the exact theory and from experiments. Comparison is shown in a number of figures, in which all spectral lines reflect the behavior of infinite trains of the principal types of waves. This preliminary assessment is valuable in that it shows that even with the simplest procedure for establishing the model constants the prediction of the way in which the principal waves propagate in the model material matches quite well the way in which comparable waves propagate in two phase layered materials.

2. HOMOGENEOUS MODEL OF A TWO PHASE MATERIAL

In a previous study[1], an approximate theory was developed which governs the dynamic behavior of a homogeneous, anisotropic, elastic, dispersive material. The material acts as a model of general two phase materials. The latter materials are general except that they must display a periodic character and each of the two phases must be linearly elastic. An outline of this theory is presented here; in the equations which follow, however, the thermal effects contained in [1] are disregarded.

The theory consists of two types of equations; those of linear momentum, and the constitutive equations. The equations are referred to a Cartesian coordinate system (x_1, x_2, x_3) and are appropriate when the deformations are infinitesimal.

The equations of linear momentum are

$$\frac{\partial \sigma_{ji}^1}{\partial x_j} + \rho_1 F_i^1 + \Gamma_{ij}^t(u_j^2 - u_j^1) = \rho_1 \dot{v}_i^1$$
$$\frac{\partial \sigma_{ji}^2}{\partial x_j} + \rho_2 F_i^2 + \Gamma_{ij}^t(u_j^1 - u_j^2) = \rho_2 \dot{v}_i^2. \tag{2.1}$$

The first and second of eqns (2.1) apply to the first and second phases of the composite material respectively. The numbers one and two identify the phases and will be denoted in general by α and β. For the indices i, j (= 1, 2, 3) the summation convention applies. The terms appearing in eqns (2.1) are defined as follows: $\rho_\alpha = n_\alpha \rho_\alpha{}^R$, partial masses for phases, measured per unit volume of composite; n_α, volume fraction of the αth phase with the property $\sum_{\alpha=1}^{2} n_\alpha = 1$; $\rho_\alpha{}^R$, mass density of αth phase; $v_i{}^\alpha$, average velocity components for phases; $u_i{}^\alpha$, average displacement components for phases; σ_{ij}^α, partial stress components for phases, measured per unit area of composite; $\rho_\alpha F_i{}^\alpha$, body force components for phases, measured per unit volume of composite; and Γ_{ij}^t, components of a time dependent elastodynamic operator describing the linear momentum interaction between the phases. The dot indicates partial differentiation with respect to time.

The constitutive equations, which we present here in matrix form, are

$$\begin{Bmatrix} \boldsymbol{\sigma}^1 \\ \boldsymbol{\sigma}^2 \end{Bmatrix} = \begin{bmatrix} \mathbf{C}^{11} & \mathbf{C}^{12} \\ \mathbf{C}^{21} & \mathbf{C}^{22} \end{bmatrix} \begin{Bmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \end{Bmatrix}. \tag{2.2}$$

In eqn (2.2) $\boldsymbol{\sigma}^\alpha$ and $\mathbf{e}^\alpha$ are vector representations of the stress and strain tensors

$$\begin{aligned} \boldsymbol{\sigma}^\alpha &= (\sigma_{11}^\alpha, \sigma_{22}^\alpha, \sigma_{33}^\alpha, \sigma_{12}^\alpha, \sigma_{13}^\alpha, \sigma_{23}^\alpha) \\ \mathbf{e}^\alpha &= (e_{11}^\alpha, e_{22}^\alpha, e_{33}^\alpha, 2e_{12}^\alpha, 2e_{13}^\alpha, 2e_{23}^\alpha), \end{aligned} \tag{2.3}$$

where the strain components, corresponding to the αth phase e_{ij}^α are defined by

$$e_{ij}^\alpha = \frac{1}{2}\left(\frac{\partial u_i^\alpha}{\partial x_j} + \frac{\partial u_j^\alpha}{\partial x_i}\right); \tag{2.4}$$

$\mathbf{C}^{\alpha\beta}$ is a 6×6 material coefficient matrix of the form

$$\mathbf{C}^{\alpha\beta} = \begin{bmatrix} C_{11}^{\alpha\beta} & \cdots & C_{16}^{\alpha\beta} \\ \vdots & & \vdots \\ C_{61}^{\alpha\beta} & \cdots & C_{66}^{\alpha\beta} \end{bmatrix}.$$

In Ref.[1], $\mathbf{C}^{11}$ and $\mathbf{C}^{22}$ were shown to be symmetric matrices. Also $\mathbf{C}^{21}$ is equal to the transpose of $\mathbf{C}^{12}$. It follows that the stiffness matrix is symmetric, i.e.

$$C_{mn}^{\alpha\beta} = C_{nm}^{\beta\alpha} \,(n, m = 1, \ldots 6).$$

Hence the constitutive equations for the general two phase material introduce 78 model constants into the theory. The term $\mathbf{C}^{12}$ accommodates a coupling in the equations in that it permits the state of stress of one phase to be affected by the deformation of the other. It was also shown in the earlier paper that because the strain energy function must be positive definite, the model constants must satisfy the inequality

$$\mathbf{S}^T\mathbf{C}\mathbf{S} \geq 0, \tag{2.5}$$

where $\mathbf{S}$ is an arbitrary twelve dimensional vector and $\mathbf{C}$ is the symmetric 12×12 model stiffness matrix defined by

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}^{11} & \mathbf{C}^{12} \\ \mathbf{C}^{T12} & \mathbf{C}^{22} \end{bmatrix}. \tag{2.6}$$

The inequality (2.5) is satisfied if $\mathbf{C}$ is positive definite.

3. HOMOGENEOUS MODEL FOR A TWO LAYER MATERIAL

In this section we choose a specific two-phase material, i.e. we choose a particular periodic geometry for our medium. We trace the influences that this specified geometry has on the general two phase theory, and modify the theory to this geometry. We will find that changes are required in both the constitutive equations and the equations of linear momentum.

The material which we study is one in which the two materials alternate in plane layers. The materials are called phases and each is identified by so called "phase constants". Each phase is isotropically elastic, so Lamé's constants μ_α and λ_α ($\alpha = 1$ or 2) and the mass densities $\rho_\alpha{}^R$ are the phase constants. The two layers have thicknesses $2h_1$ and $2h_2$ respectively.

In what follows the laminated composite is referred to a Cartesian coordinate system (x_1, x_2, x_3) in which the x_2 axis is perpendicular to the planes of layering (see Fig. 1). With respect to this reference frame the composite displays hexagonal symmetry, in which x_2 is the axis of symmetry. We first study the constitutive equations.

(a) *Constitutive equations*

The hexagonal symmetry resulting from layered periodicity results in a reduction in the number of model constants appearing in the constitutive equations. The stiffness matrices have the forms,

$$\mathbf{C}^{\alpha\beta} = \begin{bmatrix} C_{11}^{\alpha\beta} & C_{12}^{\alpha\beta} & C_{13}^{\alpha\beta} & 0 & 0 & 0 \\ C_{12}^{\alpha\beta} & C_{22}^{\alpha\beta} & C_{12}^{\alpha\beta} & 0 & 0 & 0 \\ C_{13}^{\alpha\beta} & C_{12}^{\alpha\beta} & C_{11}^{\alpha\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^{\alpha\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}^{\alpha\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44}^{\alpha\beta} \end{bmatrix} (\text{for } \alpha = \beta), \tag{3.1}$$

where

$$C_{55}^{\alpha\beta} = \frac{1}{2}(C_{11}^{\alpha\beta} - C_{13}^{\alpha\beta})$$

and

$$\mathbf{C}^{12} = \begin{bmatrix} C_{11}^{12} & C_{12}^{12} & C_{11}^{12} & 0 & 0 & 0 \\ C_{21}^{12} & C_{22}^{12} & C_{21}^{12} & 0 & 0 & 0 \\ C_{11}^{12} & C_{12}^{12} & C_{11}^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44}^{12} \end{bmatrix}. \tag{3.2}$$

Examination of eqns (3.1) and (3.2) shows that the number of independent constants has been reduced to fifteen. This reduction results from symmetry and also from recognizing that $C_{55}^{12} = 0$. The latter is obtained by integrating the constitutive relation $\overset{*}{\sigma}{}_{13}^{\alpha} = 2\mu_\alpha \overset{*}{e}{}_{13}^{\alpha}$ (where $\overset{*}{\sigma}{}_{13}^{\alpha}$ and $\overset{*}{e}{}_{13}^{\alpha}$ are the actual shear stress and strain distributions for the α phase) over the thickness of the α phase.

(b) *Equations of linear momentum*

In formulating the linear momentum equations for the general two phase theory, interaction between the phases was accounted for by the term Γ_{ij}', which we called an elastodynamic operator. With the hexagonal symmetry of the layered material, the operator has the form

$$(\Gamma_{ij}') = \begin{bmatrix} \Gamma_1' & 0 & 0 \\ 0 & \Gamma_2' & 0 \\ 0 & 0 & \Gamma_1' \end{bmatrix}. \tag{3.3}$$

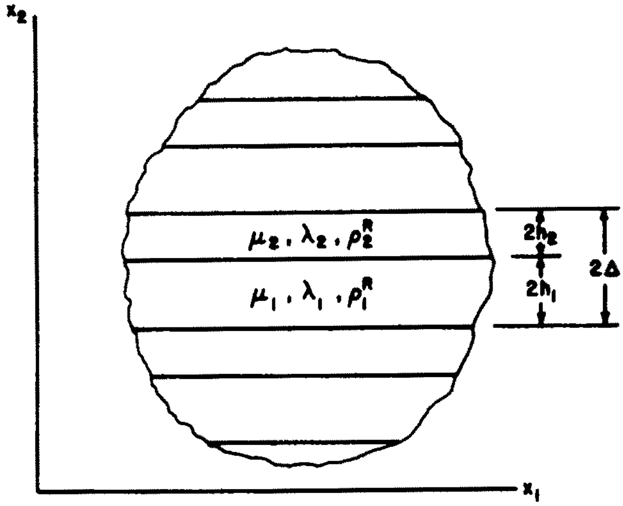

Fig. 1. A layered composite material.

The physical meaning of the components is derived from a study of eqns (2.1). The first, $\Gamma_1{}'$, describes the linear momentum interaction parallel to layering caused by the shear stresses at the interfaces, $\Gamma_2{}'$ accounts for interaction normal to layering caused by the normal, interface stresses. To gain insight into the forms of these two operators we analyze a micro model of the layered material. The analysis, which is based on assumed deformation modes for the phases, was previously used by Biot[2, 3] for constructing a viscodynamic operator for fluid filled porous media.

Micro model analysis consists of a study of a unit cell of the composite material. The unit cell consists of one layer of the first constituent bounded by two half layers of the second (see Fig. 2). We refer the cell to a Cartesian coordinate system (x_1, x_2, x_3) so that the x_1–x_3 plane coincides with the mid-plane of the first constituent and the x_2 axis is perpendicular to the layers. The analysis is approximate in that a number of simplifying assumptions are made pertaining to the state variables involved. First, the variables are assumed to depend only on the thickness variable and time, so that the displacement, for example, $\overset{*}{u}_i{}^\alpha = \overset{*}{u}_i{}^\alpha(x_2, t)$, where the asterisk is used to denote actual distributions. In the general micro model theory, the same quantities are assumed to be distributed symmetrically about the midplane of each layer. If this simple theory proves to be inadequate, we would have to employ micromorphic mixture theory which accommodates antisymmetric distributions. The results of the theory are expressed in terms of the average values of the variables over the thickness. The average value of the variable $\overset{*}{u}_i{}^\alpha$ is denoted by $u_i{}^\alpha$ (no asterisk).

The derivation of expressions for $\Gamma_1{}'$ and Γ_2' begins by substituting the displacements

$$\overset{*}{u}_1{}^\alpha = \overset{*}{u}_i{}^\alpha(x_2, t) \tag{3.4}$$

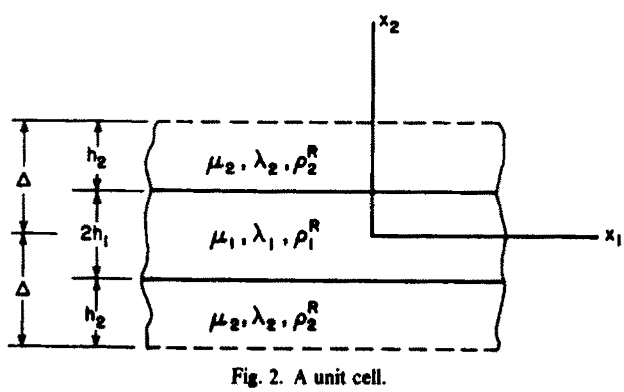

Fig. 2. A unit cell.

into the field equations of elasticity. The restricted space dependency leads to the equations

$$\mu_\alpha \frac{\partial^2 \overset{*}{u}_1{}^\alpha}{\partial x_2{}^2} + \overset{*}{F}_1{}^\alpha = \rho_\alpha{}^R \ddot{\overset{*}{u}}_1{}^\alpha$$
$$(2\mu_\alpha + \lambda_\alpha)\frac{\partial^2 \overset{*}{u}_2{}^\alpha}{\partial x_2{}^2} + \overset{*}{F}_2{}^\alpha = \rho_\alpha{}^R \ddot{\overset{*}{u}}_2{}^\alpha \tag{3.5}$$
$$\mu_\alpha \frac{\partial^2 \overset{*}{u}_3{}^\alpha}{\partial x_2{}^2} + \overset{*}{F}_3{}^\alpha = \rho_\alpha{}^R \ddot{\overset{*}{u}}_3{}^\alpha,$$

where $\alpha = 1, 2$ represents quantities in layers 1 and 2 respectively; and $\overset{*}{F}_i{}^\alpha$ are the components of the body force.

In what follows, we develop an expression for $\Gamma_1{}'$; the method is the same for $\Gamma_2{}'$. The form of $\Gamma_1{}'$ is governed by the interface shear stresses. For the assumed displacement field, these stresses are related only to the displacement parallel to the layers and the relationships are

$$\overset{*}{\sigma}{}^\alpha_{21} = \mu_\alpha \frac{\partial \overset{*}{u}_1{}^\alpha}{\partial x_2}; \; \overset{*}{\sigma}{}^\alpha_{23} = \mu_\alpha \frac{\partial \overset{*}{u}_3{}^\alpha}{\partial x_2}. \tag{3.6}$$

In our study of $\Gamma_1{}'$, we may pursue either $\overset{*}{u}_1{}^\alpha$ or $\overset{*}{u}_3{}^\alpha$ as both will lead to the same expression. We choose the first, which is governed by the first of eqns (3.5), and continue by taking the Laplace transform of each side of this equation. The result of this operation is the equation

$$\frac{d^2 \overset{*}{u}_1{}^{\alpha L}}{dx_2{}^2} - m_\alpha{}^2 \overset{*}{u}_1{}^{\alpha L} = \frac{-\overset{*}{F}_1{}^{\alpha L}}{\mu_\alpha}, \tag{3.7}$$

where the superscript L denotes the Laplace transform of the quantity and

$$m_\alpha{}^2 = \frac{\rho_\alpha{}^R}{\mu_\alpha} p^2$$

in which p is the Laplace transform parameter.

In the integration of eqn (3.7) we exploit the symmetry of $\overset{*}{u}_1{}^\alpha$ and $\overset{*}{F}_1{}^\alpha$, already discussed, and further we make the same assumption as made by Biot[2], that $\overset{*}{F}_1{}^\alpha$ is a constant. Integration leads to the equation

$$\overset{*}{u}_1{}^{\alpha L} = A_\alpha \sinh m_\alpha x_2 + B_\alpha \cosh m_\alpha x_2 + G_\alpha, \tag{3.8}$$

where

$$G_\alpha = \frac{\overset{*}{F}_1{}^{\alpha L}}{\mu_\alpha m_\alpha{}^2}. \tag{3.9}$$

The four integration constants, A_α and B_α, appearing in eqn (3.8), can be determined by using the continuity condition at the interface $x_2 = h_1$,

$$(\overset{*}{u}_1{}^1) = (\overset{*}{u}_1{}^2); \qquad (\overset{*}{\sigma}{}^1_{21}) = (\overset{*}{\sigma}{}^2_{21}), \tag{3.10}$$

and the conditions at the mid-planes $x_2 = 0$ and $x_2 = \Delta = h_1 + h_2$

$$(\overset{*}{\sigma}{}^1_{21})_{x_2=0} = 0; \qquad (\overset{*}{\sigma}{}^2_{21})_{x_2=\Delta} = 0 \tag{3.11}$$

which are imposed by the symmetry of $\overset{*}{u}_1{}^\alpha$. After some manipulation, the final form of $\overset{*}{u}_1{}^{\alpha L}$ is found to be

$$\overset{*}{u}_1{}^{1L} = \frac{C}{D}(G_1 - G_2)\cosh m_1 x_2 + G_1 \tag{3.12}$$

$$\overset{*}{u}_1{}^{2L} = \frac{1}{D}(G_1 - G_2)(\cosh m_2 x_2 - \tanh m_2\Delta \sinh m_2 x_2) + G_2,$$

where

$$\begin{aligned} C &= \frac{\mu_2 m_2}{\mu_1 m_1 \sinh m_1 h_1}(\sinh m_2 h_1 - \tanh m_2\Delta \cosh m_2 h_1) \\ D &= (F-C)\cosh m_1 h_1 \\ F &= \frac{1}{\cosh m_1 h_1}(\cosh m_2 h_1 - \tanh m_2\Delta \sinh m_2 h_1). \end{aligned} \tag{3.13}$$

To determine the interface shear stress we substitute eqns (3.12) into eqns (3.6), and obtain

$$(\overset{*}{\sigma}{}_{21}^{1L})_{x_2=h_1} = (\overset{*}{\sigma}{}_{21}^{2L})_{x_2=h_1} = \frac{C}{D}\mu_1 m_1(G_1 - G_2)\sinh m_1 h_1. \tag{3.14}$$

Our objective is to relate the interface shear stress to the difference of the average horizontal displacements. We formulate the averages

$$u_1{}^{1L} = \frac{1}{h_1}\int_0^{h_1} \overset{*}{u}_1{}^{1L}\,dx_2; \qquad u_1{}^{2L} = \frac{1}{h_2}\int_{h_1}^{\Delta} \overset{*}{u}_1{}^{2L}\,dx_2, \tag{3.15}$$

using $\overset{*}{u}_1{}^{\alpha L}$ from eqns (3.12) and obtain an expression for their difference

$$u_1{}^{2L} - u_1{}^{1L} = -(G_1 - G_2)\left[\frac{C}{D}\left(\frac{m_1\mu_1}{m_2\mu_2}\frac{1}{m_2 h_2} + \frac{1}{m_1 h_1}\right)\sinh m_1 h_1 + 1\right]. \tag{3.16}$$

By eliminating the factor $(G_1 - G_2)$ between eqns (3.14) and (3.16), after some manipulation, we get

$$(\overset{*}{\sigma}{}_{21}^{1L})_{x_2=h_1} = (\overset{*}{\sigma}{}_{21}^{2L})_{x_2=h_1} = \frac{3r_1 r_2}{r\Delta} L_1(\kappa_1{}^1, \kappa_2{}^1)(u_1{}^{2L} - u_1{}^{1L}), \tag{3.17}$$

where

$$L_1(\kappa_1{}^1, \kappa_2{}^1) = \frac{r}{3}\frac{\kappa_1{}^1\kappa_2{}^1 \sinh \kappa_1{}^1 \sinh \kappa_2{}^1}{r_1\kappa_1{}^1 \sinh \kappa_1{}^1 \cosh \kappa_2{}^1 + r_2\kappa_2{}^1 \cosh \kappa_1{}^1 \sinh \kappa_2{}^1 - \left(r_1\dfrac{\kappa_1{}^1}{\kappa_2{}^1} + r_2\dfrac{\kappa_2{}^1}{\kappa_1{}^1}\right)\sinh \kappa_1{}^1 \sinh \kappa_2{}^1} \tag{3.18}$$

and the arguments $\kappa_1{}^1$ and $\kappa_2{}^1$ are defined by

$$\kappa_\alpha{}^1 = \Delta\sqrt{\left(\frac{\rho_\alpha}{r_\alpha}\right)}\,p, \tag{3.19}$$

and

$$r_\alpha = \frac{\mu_\alpha}{n_\alpha}; \qquad r = \sum_{\alpha=1}^{2} r_\alpha, \tag{3.20}$$

where $n_\alpha = h_\alpha/\Delta$.

Even though eqn (3.17) is written in the Laplace transform domain it can be interpreted from operational calculus as given in the real time domain provided that the Laplace transform parameter p is replaced by the operator $(\partial/\partial t)$. In our analysis we adopt this interpretation and accordingly drop the superscript L in eqn (3.17). In this case $L_1(\kappa_1{}^1, \kappa_2{}^1)$ in eqn (3.18) becomes an operator relating the relative average horizontal displacement to the interface shear stress. It should be noted that the operator $L_1(\kappa_1{}^1, \kappa_2{}^1)$ is normalized so that $L_1(0,0) = 1$.

We now turn our attention to relating the elastodynamic operator $\Gamma_1{}'$ to the operator L_1. We first note that the x_1 component of the linear momentum interaction $M_1{}^1$ acting on the first constituent and defined per unit volume of composite is related to the interface shear stress by

$$M_1{}^1 = \frac{(\overset{*}{\sigma}{}_{21}^1)_{x_2=h_1}}{\Delta}. \tag{3.21}$$

On the other hand, we see from eqns (2.1) and (3.3) that in our general formulation the same interaction is given by

$$M_1{}^1 = \Gamma_1{}'(u_1{}^2 - u_1{}^1). \tag{3.22}$$

Considering eqn (3.17), and comparing eqns (3.21) and (3.22) we finally obtain the form of the horizontal component of the elastodynamic operator as

$$\Gamma_1{}' = K_1 L_1(\kappa_1{}^1, \kappa_2{}^1), \tag{3.23}$$

where the constant K_1 is given by

$$K_1 = \frac{3r_1 r_2}{\Delta^2 r}. \tag{3.24}$$

The operator $\Gamma_2{}'$, which governs the vertical linear momentum interaction, can be obtained in a similar way. In the derivation of this operator, the terms $\overset{*}{u}_1{}^\alpha$ and $\overset{*}{\sigma}{}_{21}^\alpha$ appearing in eqns (3.10) and (3.11) must be replaced by $\overset{*}{u}_2{}^\alpha$ and $\overset{*}{\sigma}{}_{22}^\alpha$, respectively. The result is

$$\Gamma_2{}' = K_2 L_2(\kappa_1{}^2, \kappa_2{}^2) \tag{3.25}$$

where the constant K_2 and the operator $L_2(\kappa_1{}^2, \kappa_2{}^2)$ can again be defined by eqns (3.24) and (3.18), respectively, if the terms $(r_1, r_2, r, \kappa_1{}^1, \kappa_2{}^1)$ in these equations are replaced by $(E_1, E_2, E, \kappa_1{}^2, \kappa_2{}^2)$, respectively, where

$$E_\alpha = \frac{2\mu_\alpha + \lambda_\alpha}{n_\alpha}; \qquad E = \sum_{\alpha=1}^{2} E_\alpha, \tag{3.26}$$

and

$$\kappa_\alpha{}^2 = \Delta \sqrt{\left(\frac{\rho_\alpha}{E_\alpha}\right)} \quad p. \tag{3.27}$$

We recognize at this point in the development that the elastodynamic operators will make the equations of linear momentum too complicated for practical use. Accordingly, we replace each of the operators by an approximation derived by expanding the operators $L_1(\kappa_1{}^1, \kappa_2{}^1)$ and $L_2(\kappa_1{}^2, \kappa_2{}^2)$, defined by eqn (3.18), in power series expansions of their arguments, and retain the first three terms. This procedure yields

$$\begin{aligned} \Gamma_1{}' &= K_1 + q_1 \frac{\partial^2}{\partial t^2} \\ \Gamma_2{}' &= K_2 + q_2 \frac{\partial^2}{\partial t^2} \end{aligned} \tag{3.28}$$

where

$$q_1 = \frac{\rho_1 r_2^2 + \rho_2 r_1^2}{5r^2}; \qquad q_2 = \frac{\rho_1 E_2^2 + \rho_2 E_1^2}{5E^2}. \tag{3.29}$$

Here we first note than eqn (3.28) is valid when the Laplace transform parameter is small. This means that eqn (3.28) represents an asymptotic form for the operators valid for large times. Secondly, we note that eqn (3.28) implies that the linear momentum interaction involves the constituent relative acceleration. This violates the principle of material indifference because the constituent relative acceleration is not an objective quantity. This point is discussed in [4] where the author states that this violation can be disregarded in linear theories.

When the expressions for Γ_1' and Γ_2' are introduced into the equations for linear momentum, eqns (2.1), we obtain the approximate form of these equations

$$\begin{aligned} \frac{\partial \sigma_{ji}^1}{\partial x_j} + \rho_1 F_i^1 + K_{ij}(u_j^2 - u_j^1) &= m_{ij}^1 \dot{v}_j^1 - q_{ij}\dot{v}_j^2 \\ \frac{\partial \sigma_{ji}^2}{\partial x_j} + \rho_2 F_i^2 + K_{ij}(u_j^1 - u_j^2) &= -q_{ij}\dot{v}_j^1 + m_{ij}^2 \dot{v}_j^2, \end{aligned} \tag{3.30}$$

where

$$(K_{ij}) = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_1 \end{bmatrix}; \qquad (q_{ij}) = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_1 \end{bmatrix}; \tag{3.31}$$

$$(m_{ij}^\alpha) = \begin{bmatrix} (\rho_\alpha + q_1) & 0 & 0 \\ 0 & (\rho_\alpha + q_2) & 0 \\ 0 & 0 & (\rho_\alpha + q_1) \end{bmatrix}. \tag{3.31}$$

Equations (3.30) along with the constitutive equations, eqns (2.2), (3.1), (3.2) make up the model theory governing the homogeneous material which replaces the two layer material. It remains to establish the nineteen model constants of the theory in terms of the mechanical properties μ_α, λ_α and ρ_α^R of each of the two constituents.

4. COMMENTS ON THE THEORY

Before establishing equations giving the model constants in terms of the phase constants, some comments on the theory are in order. There has already been work towards developing the same kind of theory. For example, Bedford and Stern [5] using a quasi-static analysis, found the same expression for K_1 (eqn 3.24). They proposed a special mixture theory for a layered composite valid only for dilatational waves propagating parallel to the layers. Their theory takes into account only the first term in the expansion of Γ_1' and neglects the coupling in the stress–strain relations.

In the development of the equations of linear momentum in Section 3, the components of the elastodynamic operator Γ_{ij}', derived using a micro model analysis, are replaced by approximate forms. One might suspect that this step could make the resulting theory valid only when the relative phase displacements vary slowly with changes in time. This restriction would not be real if the frequency range of the theory accommodates the first nonzero cut-off frequencies in spectra representing the behavior of dilatational and shear waves propagating in the principal directions. Fortunately, this is the case for the present theory. In fact the operators contain model constants that can be adjusted to make these cut-off frequencies from the exact and model theories match. In the section which follows, the constants are used for this purpose. In the last section on the assessment of the theory we show convincingly that the approximations used for the elastodynamic operator are appropriate.

We end this section with two general remarks on the theory developed in Section 3. The first is that we feel the equations of linear momentum (eqns 3.30), even though they were developed for a layered two phase material, are appropriate for all periodic, two phase materials. We will use these same equations, for example, when developing a theory to represent a homogeneous model for masonry walls, which have vertical layers in addition to horizontal. The difference between the masonry model and that in which vertical layering is neglected will be reflected in the forms for K_{ij} and q_{ij} and the values of the constants in the matrices.

The last comment has to do with the model constants K_1, K_2, q_1 and q_2. We have already found these in terms of the phase constants as given by eqns (3.24) and (3.29). These equations are found by using micro model analysis which is based on assumed deformation modes for the phases. It might well be that in producing a model that will behave like its prototype it is more appropriate to assign other values to the constants. It seems sensible when starting the process of establishing the set of model constants, to abandon the values of K_1, K_2, q_1 and q_2 established by eqns (3.24) and (3.29) and have them unassigned as are the remaining fifteen. Thus the two steps, the derivation of the forms of the equations of the model and the establishment of the model constants, are clearly separated.

5. EVALUATION OF THE CONSTANTS

The theory just completed describes a homogeneous, dispersive material with hexagonal symmetry, that will be used to act as a model or replacement of a two phase material composed by alternating plane layers, each made up of an isotropically elastic material. The effectiveness of the model can be judged only when it has been decided what behavior of the prototype it is that we wish to mimic. In this study we will use the new material to predict the dynamic behavior of the layered material. We have at our disposal nineteen constants that we can use to this end.

We can establish the constants by matching the dynamic behaviors of the model and prototype but the possible matching phenomena are almost without limit, certainly far beyond nineteen, so that there is no unique set of constants or no unique material to be established. By the dynamic behavior of the layered material we could mean dilatational waves parallel to the layers, or shear waves normal to the layers or other variations; we could be referring to infinite trains of waves or transient wave behavior of each type; and the behavior we are considering could be that predicted by exact theories of layered materials, behavior of the model, or behavior observed from experiments.

Our choice of a layered material as a special case of a two phase, periodic material was made partly because a layered material is close to masonry but, equally important, the choice was made because there is a rich supply of information about the dynamic behavior of layered materials. We have at hand spectra developed from the exact theory for various types of trains of waves, which can be adapted to any layer properties; experimental data describing the behavior of both trains of waves and transient waves in several different two layer materials.

We can say unequivocally that the best set of constants would be those found using system identification to match some experimental behavior, either steady state or transient. If we used system identification we would have to decide at the beginning what few among the variety of experimental behaviors it is that we choose to match. If it is the behavior of infinite trains of waves in a variety of directions as reflected in spectral lines representing phase velocity vs wave number then a criterion function would be chosen which would represent the accumulation of least squared errors between experimental points and model points on the spectra over whatever length of spectral lines we choose. An optimization algorithm would then be constructed from which the nineteen constants are established that would minimize the criterion function. We have had considerable experience with system identification, enough to know, that with nineteen parameters involved, the problem in using this method would be formidable. Accordingly, we leave this to a future study.

The method we use here to establish the nineteen constants in terms of the layer properties, (which turns out to be successful) is a mixture of micro model analysis and matching of specific properties of spectra from the exact theory reflecting the behavior of infinite trains of particular types of waves. The method described in what follows is, we think, the simplest.

As we will be matching spectra for infinite trains of waves, we require the velocity equation, and subsequently the frequency equation, which govern infinite trains of waves travelling in our model material. To this end we adopt the trial solution

$$u_j^\alpha = A_j^\alpha \, e^{ik(e_p x_p - ct)} \tag{5.1}$$

in which A_j^α are the amplitudes, e is the unit normal denoting the direction of propagation, k is the wave number and c the phase velocity. When we substitute (5.1) into the constitutive relations (eqns 2.2), and the equations of linear momentum (eqns 3.30), and use the strain-displacement relations (eqn 2.4), the condition for which a nontrivial solution will exist gives the equation

$$[(\rho_1\rho_2 + \rho q)k^2]c^4 - [(\rho_1 S_{22} + \rho_2 S_{11} + qS)k^2 + \rho K]c^2 + [(S_{11}S_{22} - S_{12}S_{12})k^2 + SK] = 0 \tag{5.2}$$

for waves propagating in the x_1 and x_2 directions.

Equation (5.2) relates the phase velocity c to the wave number k and it should be noted that for each k there are two phase velocities leading to two spectral lines on the k–c plane. Equation (5.2) will be used in the next section to trace out the spectral lines for a variety of types of waves, but in this section we are primarily interested in the case where k is zero or approaches zero. Study of eqn (5.2) shows that as k approaches zero, one value of c approaches infinity, and the other approaches the cut-off velocity

$$c^2 = \frac{S}{\rho} \tag{5.3}$$

where

$$\rho = \sum_{\alpha=1}^{2} \rho_\alpha; \qquad S = \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} S_{\alpha\beta}.$$

The values of $S_{\alpha\beta}$, K and q appearing in eqns (5.2) and (5.3) differ for each type of principal wave. They are

for dilatational waves in the x_1 direction

$$S_{\alpha\beta} = C_{11}^{\alpha\beta}; \qquad K = K_1; \qquad q = q_1 \tag{5.4}$$

for S_v waves in the x_1 direction

$$S_{\alpha\beta} = C_{44}^{\alpha\beta}; \qquad K = K_2; \qquad q = q_2 \tag{5.5}$$

for S_H waves in the x_1 direction

$$S_{\alpha\beta} = C_{55}^{\alpha\beta}; \qquad K = K_1; \qquad q = q_1 \tag{5.6}$$

for dilatational waves in the x_2 direction

$$S_{\alpha\beta} = C_{22}^{\alpha\beta}; \qquad K = K_2; \qquad q = q_2 \tag{5.7}$$

for shear waves in the x_2 direction

$$S_{\alpha\beta} = C_{44}^{\alpha\beta}; \qquad K = K_1; \qquad q = q_1. \tag{5.8}$$

The phase velocity of the second branch goes to infinity as the wave number approaches zero, so additional insight can be gained by studying the frequency equation. This is easily obtained from eqn (5.2) when we recognize that the circular frequency ω is related to k and c according to $\omega = kc$. The frequency equation becomes

$$[(\rho_1\rho_2 + \rho q)]\omega^4 - [(\rho_1 S_{22} + \rho_2 S_{11} + qS)k^2 + \rho K]\omega^2 + [(S_{11}S_{22} - S_{12}S_{12})k^4 + SKk^2] = 0. \tag{5.9}$$

When $k = 0$, the equation becomes

$$(\rho_1\rho_2 + \rho q)\omega^4 - \rho K\omega^2 = 0 \tag{5.10}$$

so that the lowest spectral line emanates from the origin of the $\omega - k$ plane, and the second has the cut-off frequency

$$\omega_\alpha^{\,2} = \frac{\rho K_\alpha}{(\rho_1 \rho_2 + \rho q_\alpha)}. \tag{5.11}$$

The frequency ω_1 is the cut-off for P and S_H waves propagating in the x_1 direction, and S_v waves propagating in the x_2 direction; ω_2 for S_v waves in the x_1 direction and P waves in the x_2 direction.

We now are in a position to begin the process of establishing equations that relate the model constants to the properties of each of the two layers in the prototype.

Determination of K_α and q_α

We first establish equations from which K_1, K_2, q_1 and q_2, the constants appearing in the equations of linear momentum, can be found. In the previous section while developing the form for the elastodynamic operator using micro model analysis, we found values for K_1 and K_2. These values were temporarily abandoned to leave all constants free for evaluation by other matching, but now we again adopt these equations. Accordingly, we establish K_1 and K_2 from eqn (3.24).

On the other hand, we do not use our previous equations to establish q_1 and q_2. These constants determine directly the cut-off frequencies of infinite trains of waves, so we adjust them so that the cut-off frequencies of P and S_v waves in the x_2 direction predicted by the model are the same as those predicted by the exact theory.

The cut-off frequencies $\bar{\omega}_1$ and $\bar{\omega}_2$ from the exact theory are the lowest nonzero roots respectively of the equations

$$\begin{aligned} \cos\xi_1 \cos\xi_2 - \frac{1+p_1^{\,2}}{2p_1}\sin\xi_1 \sin\xi_2 &= 1 \\ \cos\eta_1 \cos\eta_2 - \frac{1+p_2^{\,2}}{2p_2}\sin\eta_1 \sin\eta_2 &= 1. \end{aligned} \tag{5.12}$$

In eqns (5.12)

$$\begin{aligned} &\xi_\alpha = 2h_\alpha \bar{\omega}_1 / C_T^{\,\alpha}; && \eta_\alpha = 2h_\alpha \bar{\omega}_2 / C_L^{\,\alpha} \\ &C_T^{\,\alpha} = \sqrt{\left(\frac{\mu_\alpha}{\rho_\alpha^{\,R}}\right)}; && p_1 = \frac{\mu_1}{\mu_2}\frac{C_T^{\,2}}{C_T^{\,1}} \\ &C_L^{\,\alpha} = \sqrt{\left(\frac{\lambda_\alpha + 2\mu_\alpha}{\rho_\alpha^{\,R}}\right)}; && p_2 = \frac{\mu_1 + 2\mu_1}{\lambda_2 + 2\mu_2}\frac{C_L^{\,2}}{C_L^{\,1}}. \end{aligned} \tag{5.13}$$

The constants q_1 and q_2 are obtained by equating the $\omega_\alpha^{\,2}$ in eqn (5.11) to the $\bar{\omega}_\alpha^{\,2}$ obtained from eqns (5.12).

Determination of $C_{11}^{\alpha\beta}$, $C_{12}^{\alpha\beta}$ and $C_{13}^{\alpha\beta}$

All of the equations establishing these constants are derived using micro model analysis. We present the derivation of the equations from which the $C_{11}^{\alpha\beta}$ are found, and then state the comparable equations for the $C_{12}^{\alpha\beta}$ and $C_{13}^{\alpha\beta}$.

We assume a quasi-longitudinal deformation state parallel to layering (see Fig. 2), so that distance between the mid-planes of layers remain unchanged. If the lateral expansion (or contraction) of the first constituent is δ, that of the second will be $(-\delta)$.

When we average the constitutive relations for $\overset{*}{\sigma}{}_{11}^{\alpha}$ and $\overset{*}{\sigma}{}_{22}^{\alpha}$

$$\begin{aligned} \overset{*}{\sigma}{}_{11}^{\alpha} &= (\lambda_\alpha + 2\mu_\alpha)\overset{*}{e}{}_{11}^{\alpha} + \lambda_\alpha \overset{*}{e}{}_{22}^{\alpha} \\ \overset{*}{\sigma}{}_{22}^{\alpha} &= (\lambda_\alpha + 2\mu_\alpha)\overset{*}{e}{}_{22}^{\alpha} + \lambda_\alpha \overset{*}{e}{}_{11}^{\alpha} \end{aligned} \tag{5.14}$$

over the thickness of the αth constituent by using the formulas

$$\bar{\sigma}^1_{11} = \frac{1}{2h_1}\int_{-h_1}^{h_1} \overset{*}{\sigma}{}^1_{11}\, dx_2$$
$$\bar{\sigma}^2_{11} = \frac{1}{2h_2}\int_{h_1}^{\Delta+h_2} \overset{*}{\sigma}{}^2_{11}\, dx_2; \text{ etc.} \tag{5.15}$$

we obtain

$$\bar{\sigma}^1_{11} = n_1 E_1 \bar{e}^1_{11} + \lambda_1 \frac{\delta}{2h_1}$$
$$\bar{\sigma}^2_{11} = n_2 E_2 \bar{e}^2_{11} - \lambda_2 \frac{\delta}{2h_2} \tag{5.16}$$

and

$$\bar{\sigma}^1_{22} = \lambda_1 \bar{e}^1_{11} + n_1 E_1 \frac{\delta}{2h_1}$$
$$\bar{\sigma}^2_{22} = \lambda_2 \bar{e}^2_{11} - n_2 E_2 \frac{\delta}{2h_2}. \tag{5.17}$$

At the interfaces of layers, $\overset{*}{\sigma}{}^\alpha_{22}$ should be continuous. In the micro model we assume that this continuity is approximately satisfied by that of the average $\bar{\sigma}^\alpha_{22}$. By using eqns (5.17) and the continuity of $\overset{*}{\sigma}{}^\alpha_{22}$, the term δ appearing in eqns (5.16) can be eliminated. This yields

$$\bar{\sigma}^1_{11} = \left(n_1 E_1 - \frac{\lambda_1^2}{n_1 E}\right)\bar{e}^1_{11} + \frac{\lambda_1\lambda_2}{n_1 E}\bar{e}^2_{11}$$
$$\bar{\sigma}^2_{11} = \frac{\lambda_1\lambda_2}{n_2 E}\bar{e}^1_{11} + \left(n_2 E_2 - \frac{\lambda_2^2}{n_2 E}\right)\bar{e}^2_{11}. \tag{5.18}$$

When we use the equation $\sigma^\alpha_{11} = n_\alpha \bar{\sigma}^\alpha_{11}$ which relates the partial stress σ^α_{11} to the average stress $\bar{\sigma}^\alpha_{11}$ and note that $e^\alpha_{11} = \bar{e}^\alpha_{11}$, we finally obtain

$$\sigma^1_{11} = \left(n_1^2 E_1 - \frac{\lambda_1^2}{E}\right) e^1_{11} + \frac{\lambda_1\lambda_2}{E} e^2_{11}$$
$$\sigma^2_{11} = \frac{\lambda_1\lambda_2}{E} e^1_{11} + \left(n_2^2 E_2 - \frac{\lambda_2^2}{E}\right) e^2_{11}. \tag{5.19}$$

To obtain the constants $C^{\alpha\beta}_{11}$ in terms of the properties of the layers we compare the expression for σ^1_{11} and σ^2_{22} given by eqns (5.19) with those from the constitutive relations equations (2.2). The comparable stresses are equal when

$$C^{11}_{11} = n_1^2 E_1 - \frac{\lambda_1^2}{E}; \quad C^{22}_{11} = n_2^2 E_2 - \frac{\lambda_2^2}{E}; \quad C^{12}_{11} = C^{21}_{11} = \frac{\lambda_1\lambda_2}{E}. \tag{5.20}$$

The relations, eqns (5.20), were obtained previously in [6], where a mixture theory, valid only for dilatational waves propagating parallel to layering, was proposed.

Here we also note that a similar procedure to that described above was previously used by Stern and Bedford [7] to establish all of the constants of a model which they proposed by using the theory of mixtures. In that work Stern and Bedford did not take into account the coupling in the stress–strain relations which would imply that the state of stress on one phase is affected by the deformation of the other.

Using the same method as for $C^{\alpha\beta}_{11}$, we find

$$C^{11}_{12} = n_1\lambda_1\frac{E_2}{E};\quad C^{22}_{12} = n_2\lambda_2\frac{E_1}{E};\quad C^{12}_{12} = n_2\lambda_1\frac{E_2}{E}$$

$$C^{21}_{12} = C^{12}_{21} = n_1\lambda_2\frac{E_1}{E} \tag{5.21}$$

and

$$C^{11}_{13} = n_1\lambda_1 - \frac{\lambda_1^{\,2}}{E};\qquad C^{22}_{13} = n_2\lambda_2 - \frac{\lambda_2^{\,2}}{E}. \tag{5.22}$$

We note from Ref.[8] that the constants found using eqns (5.20)–(5.22) are identical with those that will allow the cut-off phase velocities of P and S_H waves in the x_1 direction from the model to match those given by the exact theory. The last analysis also gives $C^{12}_{13} = C^{21}_{13} = (\lambda_1\lambda_2/E)$, but each of these is equal to C^{12}_{11}, already established, so these equalities do not provide independent information.

It only remains to establish $C^{\alpha\beta}_{22}$ and $C^{\alpha\beta}_{44}$, and we choose to find these sets of constants by adjusting the cut-off and asymptotic phase velocities of infinite trains of waves of two specific types.

Determination of $C^{\alpha\beta}_{22}$ and $C^{\alpha\beta}_{44}$

We find three independent constants C^{11}_{22}, C^{22}_{22} and C^{12}_{22} ($= C^{21}_{22}$) by matching three properties of the spectral lines representing the relationship between phase velocity and wave numbers for dilational waves travelling perpendicular to layering. A similar procedure was previously used by Mindlin and McNiven[9] who matched cut-off frequencies to establish adjustment factors as part of an approximate theory governing the vibrations of rods. We first match the cut-off velocity of the lowest branch. We already have the velocity for the model from eqns (5.3) and (5.7). The same velocity from the exact theory is

$$\bar{c}^2 = \frac{E_1E_2}{\rho E}. \tag{5.23}$$

Matching gives the equation

$$C^{11}_{22} + C^{22}_{22} + 2C^{12}_{22} = \frac{E_1E_2}{E}. \tag{5.24}$$

The other two equations are obtained by matching the asymptotic phase velocities for the lowest and second lines, respectively.

We know from the exact theory that the asymptotic phase velocity of the lowest branch is zero. When we derive this same velocity for the model from eqn (5.2) and set it equal to zero, we get

$$C^{11}_{22}C^{22}_{22} - C^{12}_{22}C^{12}_{22} = 0. \tag{5.25}$$

Finally, from the exact theory we have the expression for the fastest transmitted wave velocity.

$$c_\infty = \frac{\sqrt{\left(\frac{E_1}{\rho_1}\right)}\sqrt{\left(\frac{E_2}{\rho_2}\right)}}{\sqrt{\left(\frac{E_1}{\rho_1}\right)} + \sqrt{\left(\frac{E_2}{\rho_2}\right)}}. \tag{5.26}$$

When the velocity is equated to the second asymptotic phase velocity for the model derived from eqn (5.2), we get a third independent equation

$$\rho_1C^{22}_{22} + \rho_2C^{11}_{22} = (\rho_1\rho_2 + \rho q_2)c_\infty^{\,2} - q_2\frac{E_2E_1}{E}. \tag{5.27}$$

Comparable equations are obtained by matching the same quantities, this time for S_v waves propagating in the x_2 direction. The equations are

$$C^{11}_{44}+C^{22}_{44}+2C^{12}_{44}=\frac{r_1r_2}{r} \tag{5.28}$$

$$C^{11}_{44}C^{22}_{44}-C^{12}_{44}C^{12}_{44}=0 \tag{5.29}$$

$$\rho_1C^{22}_{44}+\rho_2C^{11}_{44}=(\rho_1\rho_2+\rho q_1)c_\infty{}^2-q_1\frac{r_1r_2}{r}. \tag{5.30}$$

In eqn (5.30)

$$c_\infty=\frac{\sqrt{\left(\frac{r_1}{\rho_1}\right)}\sqrt{\left(\frac{r_2}{\rho_2}\right)}}{\sqrt{\left(\frac{r_1}{\rho_1}\right)}+\sqrt{\left(\frac{r_2}{\rho_2}\right)}}. \tag{5.31}$$

With the set of equations established in this section, one is able to calculate the nineteen model constants for a particular layered material for which the phase constants are specified. This completes the formulation of the theory governing the model material.

6. ASSESSMENT OF THE MODEL

The final step in completing a theory is to ascertain how well it works. Here, we have available a wealth of material describing three types of dynamic behavior of various layered materials. The first is theoretical and consists of phase velocity and frequency spectra that predict the behavior of infinite trains of waves from the exact theory. The second displays the

Table 1. Properties of Sun's material

SPECIFIED LAYER PROPERTIES							
n_1	n_2	$\bar{\rho}^R_1$	$\bar{\rho}^R_2$	$\bar{\mu}_1$	$\bar{\mu}_2$	$\bar{\lambda}_1$	$\bar{\lambda}_2$
0.8	0.2	3	1	100	1	150	2.333
COMPUTED MODEL CONSTANTS							
$\bar{K}_1$	$\bar{K}_2$	$\bar{q}_1$	$\bar{q}_2$				
36.923	158.550	0.041	0.040				
$\bar{C}^{11}_{11}$	$\bar{C}^{22}_{11}$	$\bar{C}^{12}_{11}$	$\bar{C}^{11}_{12}$	$\bar{C}^{22}_{12}$	$\bar{C}^{12}_{12}$	$\bar{C}^{12}_{21}$	
230.998	0.855	0.762	5.663	0.445	1.416	1.778	
$\bar{C}^{11}_{13}$	$\bar{C}^{22}_{13}$	$\bar{C}^{11}_{22}$	$\bar{C}^{22}_{22}$	$\bar{C}^{12}_{22}$	$\bar{C}^{11}_{44}$	$\bar{C}^{22}_{44}$	$\bar{C}^{12}_{44}$
70.998	0.455	44.076	4.389	-13.908	10.686	1.158	-3.517

THE DIMENSIONLESS QUANTITIES APPEARING IN THIS TABLE ARE DEFINED BY

$$\bar{\rho}^R_\alpha=\rho^R_\alpha/\rho^R_2\ ;\ \bar{\mu}_\alpha=\mu_\alpha/\mu_2\ ;\ \bar{\lambda}_\alpha=\lambda_\alpha/\mu_2$$

$$\bar{K}_\alpha=\frac{(2h_i)^2K_\alpha}{\mu_2}\ ;\ \bar{q}_\alpha=q_\alpha/\rho^R_2\ ;\ \bar{C}^{\alpha\beta}_{ij}=C^{\alpha\beta}_{ij}/\mu_2.$$

Table 2. Properties of thornel reinforced carbon phenolic

SPECIFIED LAYER PROPERTIES							
h_1	h_2	ρ_1^R	ρ_2^R	μ_1	μ_2	λ_1	λ_2
cm		$\frac{\text{dyne-}\mu\text{sec}^2}{\text{cm}^4}$		dyne/cm^2		dyne/cm^2	
0.0032	0.0279	1.47×10^{12}	1.42×10^{12}	0.756×10^{12}	0.0662×10^{12}	0.756×10^{12}	0.114×10^{12}

COMPUTED MODEL CONSTANTS				
K_1	q_1	C_{11}^{11}	C_{11}^{22}	C_{11}^{12}
dyne/cm^4	$\frac{\text{dyne-}\mu\text{sec}^2}{\text{cm}^4}$	dyne/cm^2	dyne/cm^2	dyne/cm^2
227.213×10^{12}	0.399×10^{12}	0.210×10^{12}	0.220×10^{12}	0.0039×10^{12}

Table 3. Propert'es of boron reinforced carbon phenolic

SPECIFIED LAYER PROPERTIES							
h_1	h_2	ρ_1^R	ρ_2^R	μ_1	μ_2	λ_1	λ_2
cm		$\frac{\text{dyne-}\mu\text{sec}^2}{\text{cm}^4}$		dyne/cm^2		dyne/cm^2	
0.0052	0.026	2.37×10^{12}	1.42×10^{12}	0.951×10^{12}	0.0662×10^{12}	0.806×10^{12}	0.114×10^{12}

COMPUTED MODEL CONSTANTS				
K_1	q_1	C_{11}^{11}	C_{11}^{22}	C_{11}^{12}
dyne/cm^4	$\frac{\text{dyne-}\mu\text{sec}^2}{\text{cm}^4}$	dyne/cm^2	dyne/cm^2	dyne/cm^2
239.387×10^{12}	0.280×10^{12}	0.410×10^{12}	0.204×10^{12}	0.0055×10^{12}

same behavior from experimental observations, and the third describes transient wave behavior from experiments.

The use of this transient wave behavior to assess the approximate theory is probably the most demanding and the most satisfying, but since it requires additional theoretical development this assessment is left to a separate paper.

We compare spectra derived from the model theory with the exact theory spectra of Sun *et al.*[10] which they used to assess a first order effective stiffness theory for a layered composite. The phase constants assumed by Sun and the comparable model constants are shown in Table 1. The experimental results used here are due to Whittier *et al.*[11]. They established spectral lines representing the relationship between phase velocity and frequency for waves propagating parallel to layering. The spectral lines are for two different layered material; thornel reinforced carbon phenolic and boron reinforced carbon phenolic. The elastic properties of each of these materials are shown in Tables 2 and 3, along with their respective model constants that are needed.

Comparisons with the exact theory are displayed in Figs. 3–9. In these figures, the dimensionless wave number ξ, the phase velocity β and the normalized frequency Ω are given by

$$\xi = 2h_1k; \qquad \beta = c/(\mu_2/\rho_2^R)^{1/2}; \qquad \Omega = 2h_1\omega/(\mu_2/\rho_2^R)^{1/2}. \tag{6.1}$$

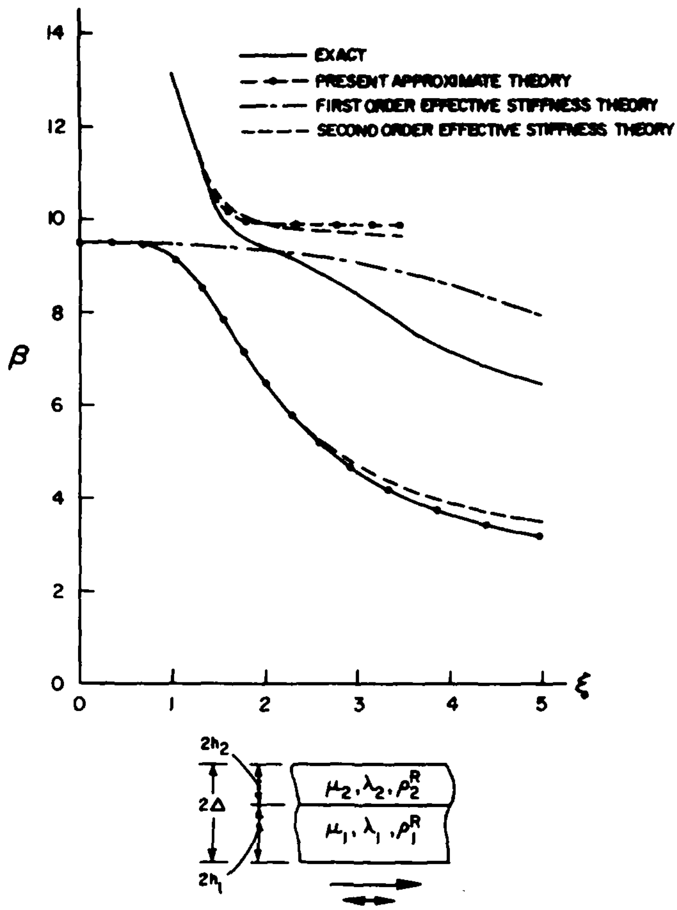

Fig. 3. Spectrum for dilatational waves propagating parallel to layering (Sun's material).

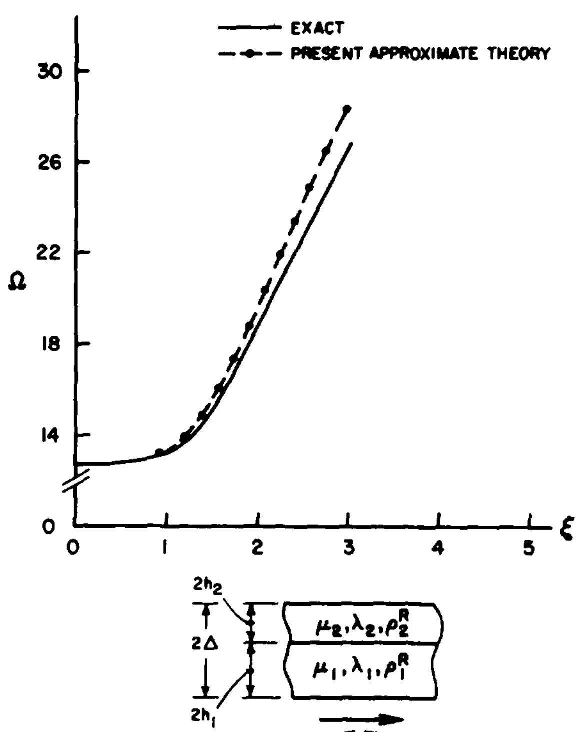

Fig. 4. The second spectral line on the frequency-wave number plane for dilatational waves propagating parallel to layering (Sun's material).

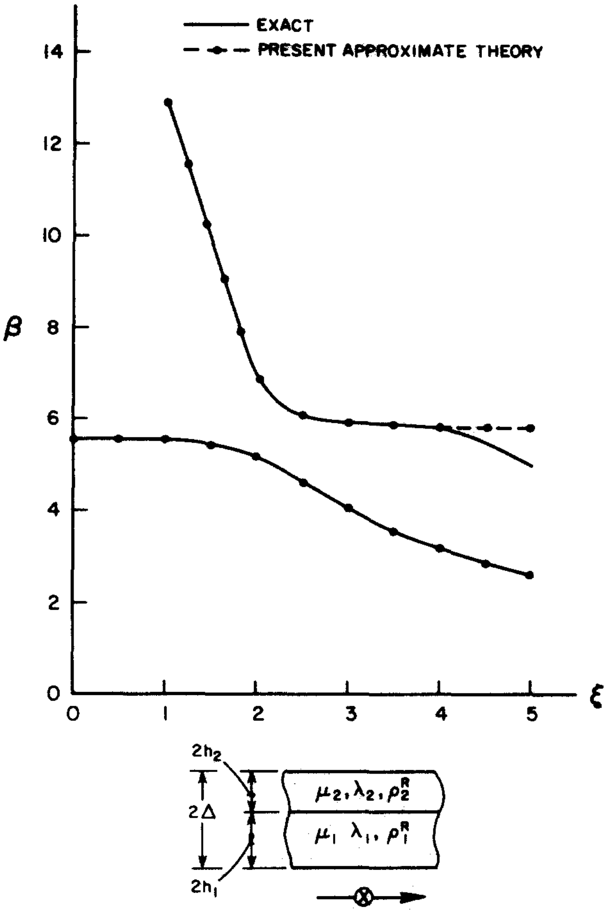

Fig. 5. Spectrum for *SH* waves propagating parallel to layering (Sun's material).

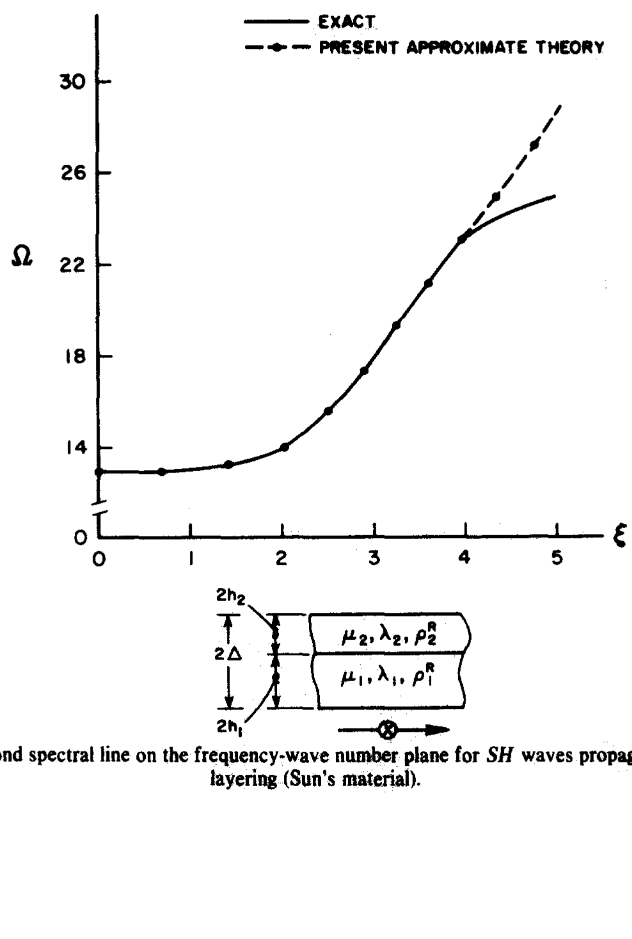

Fig. 6. The second spectral line on the frequency-wave number plane for *SH* waves propagating parallel to layering (Sun's material).

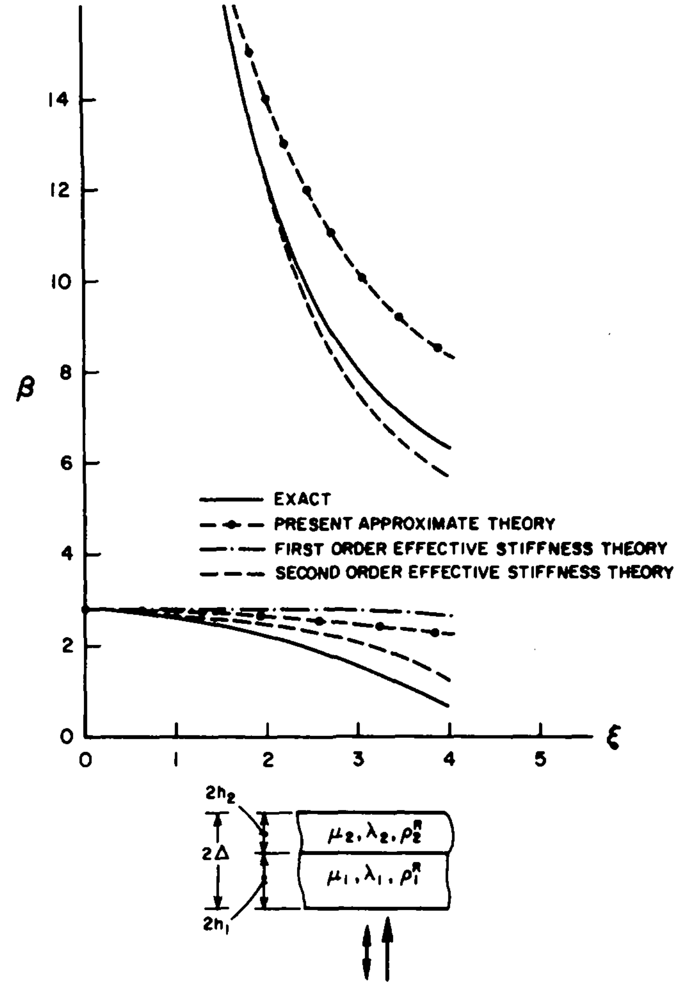

Fig. 7. Spectrum for dilatational waves propagating perpendicular to layering (Sun's material).

Figures 3–8 show comparisons between spectral lines representing phase velocities for waves travelling in principal directions. The matching is remarkably good. Figure 9 shows the same comparison for waves propagating parallel to the x_1–x_2 plane at an angle of 45° with the x_1 axis. In Figs. 3 and 7 comparison is made not only with the exact theory but with spectral lines derived from both first and second order effective stiffness theories[10, 12]. From both figures we may conclude that the theory presented here is superior to the first order theory. In Fig. 3, the present theory and the second order theory are about equally effective; while for P waves travelling in the x_2 direction shown in Fig. 7, the second order effective stiffness theory is superior to the present theory. However, it is important to point out that this preferable performance is obtained at the expense of a considerable increase in complication. Even though the dispersion introduced by the lower approximate spectral line appears to be slight, we can report that the transient wave behavior, predicted by the present theory, of P waves propagating in the x_2 direction agrees well with the behavior reported from experiments. This comparison will be presented in a separate study.

In all of these figures the spectral lines give the relationship between phase velocity and wave number. Since the second spectral line extends toward infinite phase velocity as the wave number approaches zero, very little insight into the behavior of this higher mode is gained from these figures for small wave numbers. Accordingly, for this mode, a comparison is made of spectral lines reflecting the relationship between frequency and wave number; these lines have finite cut-offs. Frequency spectra for this higher mode for P waves and S_H waves propagating parallel to layering are shown in Figs. 4 and 6, respectively.

The last two figures, Figs. 10 and 11, are devoted to thornel and boron reinforced carbon phenolic laminates, respectively. In the figures we compare the first spectral lines found using

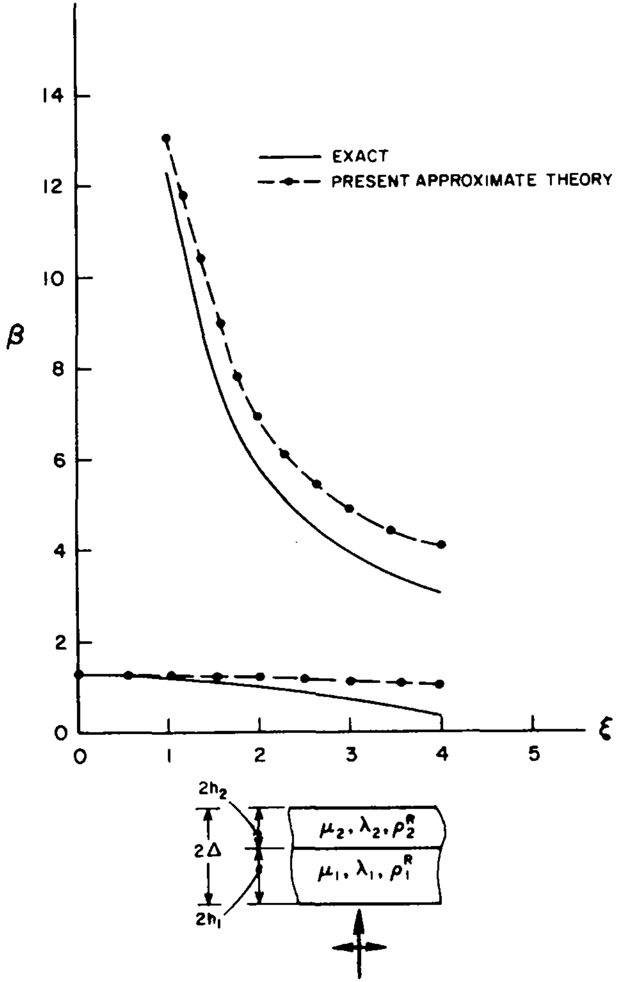

Fig. 8. Spectrum for transverse waves propagating perpendicular to layering (Sun's material).

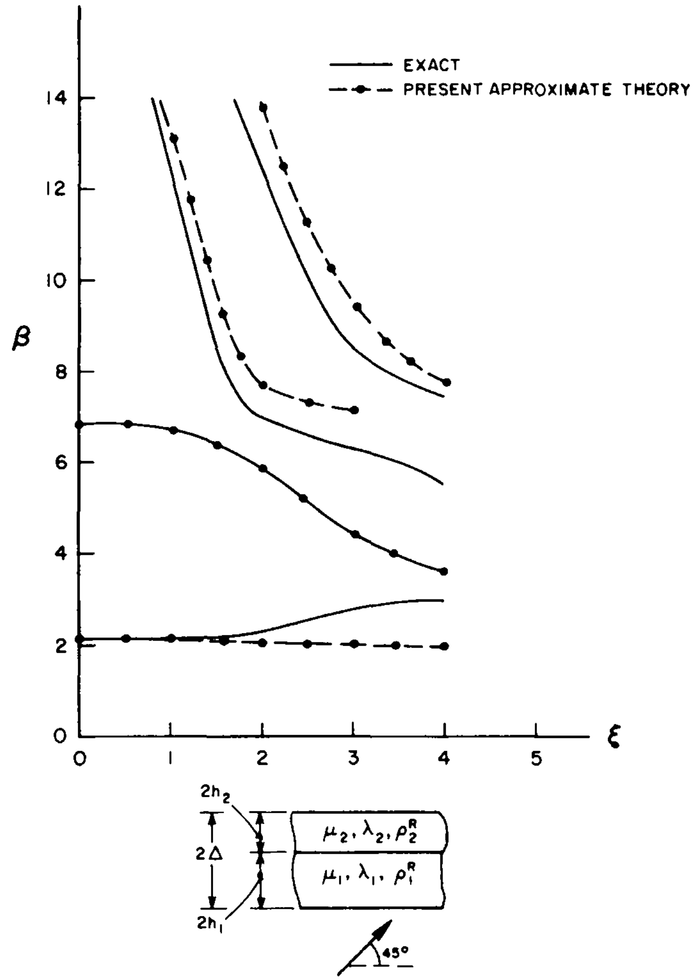

Fig. 9. Spectrum for waves propagating at an angle of 45° with the x_1 axis (Sun's material).

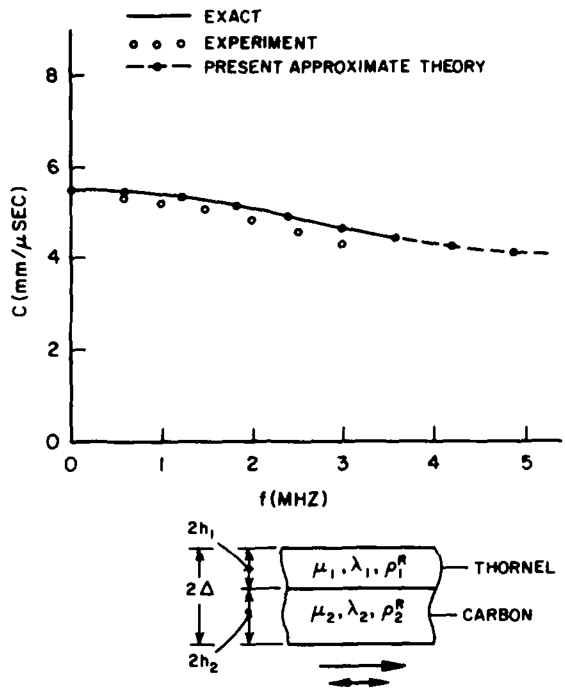

Fig. 10. First spectral line for dilatational waves propagating parallel to layering (thornel reinforced carbon phenolic).

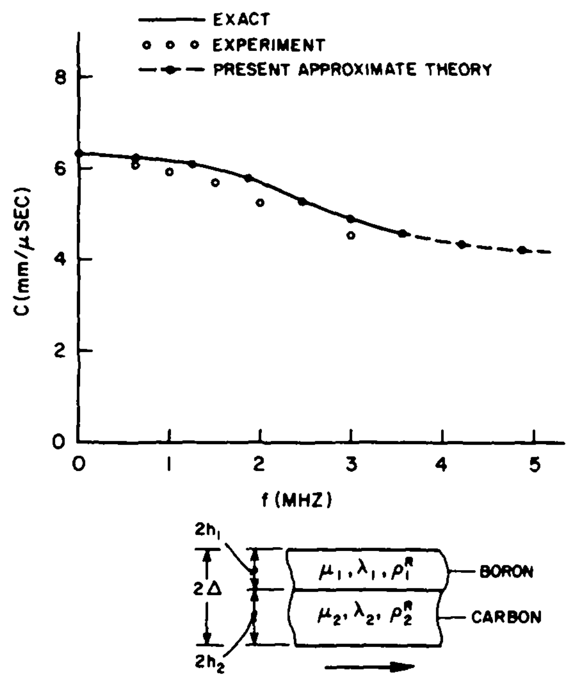

Fig. 11. First spectral line for dilatational waves propagating parallel to layering (boron reinforced carbon phenolic).

the present theory with the exact and experimenta ones for dilatational waves propagating parallel to layering. The comparisons are made on the $(f–c)$ plane, where f is the cyclic frequency, related to the angular frequency ω by $f = \omega/2\pi$. The figures indicate that agreement between experimental and theoretical data is remarkably good.

Finally, an important point should be noted. For waves propagating perpendicular to layering, the exact spectrum has a banded structure with passing and stopping bands. These bands are regions of the spectrum representing harmonic waves that propagate and attenuate respectively. The present approximate theory and others, except those proposed by Herrmann, Kaul and Delph[13, 14], do not predict the stopping bands. In these papers the authors have developed a one-dimensional approximate theory, which they call effective dispersion theory, for waves propagating perpendicular to layering only. Their theory accommodates the first stopping band and approximates quite well the two lowest spectral lines over the first two Brillouin zones. A refined three-dimensional approximate theory based on micromorphic mixture theory is being developed for two phase composites by the authors of the present work and will be reported soon. In that work it is found that the refined theory, which accomodates both the symmetric and antisymmetric displacement distributions, within phases, predicts the stopping bands for waves propagating perpendicular to layering.

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